**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part I. Calculus of variations

## Lecture 4. Special cases of the Lagrange problem

Calculus of variations consider different problems of the functional minimization. We considered before the Lagrange problem. This is the minimization problem for the integral functional with given boundary conditions, where the functional depends from the unknown function of one variable and its first derivative. This problem can be transformed to the Euler equation. This is the difficult enough second order differential for the general situation. However, its analysis can be simplified, if our problem has a special form. We determine the first integrals for solving the given problem. The fall of the body, the refraction of the light, and the brachistochrone problem are considered as examples.

### 4.1. First integrals

We considered the minimization problem for the functional



with boundary conditions. Its solution *u* satisfies the Euler equation



This is the second order differential equation that can be difficult enough for the general case. However, we could obtain easier result for special cases.

Consider the general form of the second order differential equations



**Definition 4.1.** *The function*  *is called the* ***first integral*** *of this system*, *if G is constant for all solution of the given equation.*

If we can find the first integral of the system, we transform the given second order differential equation to the first order differential equation



Therefore, if we find the first integral of the equation, then we simplify it.

We consider an application of this result. Let us have the Lagrange problem with a function *F* under the integral that does not depend from the independent variable *x*. We have the problem of minimization for the functional



**Lemma 4.1**.*Suppose the function F under the integral does not depend from the independent variable x. Then the Euler equation can be transformed to the equality*

 (4.1)

*where c is an arbitrary constant*.

**Proof**. Find the value

.

Using the Euler equation, we obtain the equality



Therefore, the equality (4.1) is true.

**Corollary 4.1.** *The function*  *is the first integral**of the system*.

The equation (4.1) has the first order. Of course, this relation is simpler than the standard Euler equation. We consider applications of this result.

### 4.2. Example

Consider the problem of minimization of the functional



The function under the integral



does not depends from the independent variable. Therefore, we can determine a solution of the problem based on its first integral. Then we have



We get



Thus, 

### 4.3. The fall of the body

We considered before the phenomenon of the fall of the body. It is described by the minimization problem for the action



where the independent variable *t* is the time, the unknown function *y* is the coordinate of the body, *m* is the mass, and *g* is the gravitational acceleration. The function



does not depends from *t.* Therefore, Lemma 4.1 is applicable here.

Find the first integral of the system



|  |
| --- |
| **Question**: *What is the physical sense of the first integral?* |

The first term at the left-hand side of this equality is the kinetic energy of the body, and the second one is its potential energy. Its sum is the total mechanic energy *E*.

Now determine the equality (4.1)



|  |
| --- |
| **Question**: *What is the physical sense of this equality?* |

The relation (4.1) has the form



By this equality, the completely energy of the body is constant. This is the ***law of energy conservation***.

### 4.4. Fermat principle and the law of the refraction of light

One of the important optical laws is the ***Fermat principle***. By the Fermat principle, the light moves by the curve that gives the minimal time of the movement. Of course, if the environment is homogeneous, then the light moves by a line. However, this phenomenon is not trivial for the non-homogeneous environments. We try to use the Fermat principle for determining the law of the refraction of the light.

We consider the movement of the light on the plane by the curve  The initial and the final points are known. Therefore, we have the boundary conditions

 (4.2)

The velocity of the movement is determine by the formula



where the function  is the way of the light. Then we have the equality



We know (see the previous lecture) that the way by the curve  can be determine by the equality



Then the previous formula can be transformed to the equality



We know the initial point  at the initial time  and the final point  for the final time  After integration we get

 (4.3)

By the Fermat principle, the curve  of real movement of the light minimizes the time (4.3).

For easiest case, the considered environment is homogeneous. Then the velocity *v* of the light is constant. Therefore, the formula (4.3) can be transformed to the equality



We considered before the problem of minimization of the functional



This is minimization problem of length of the curve . Its solution is a linear function. Therefore, we have the analogical result. Thus, the light moves by the light in the homogeneous environment.

Now we consider more interesting and more difficult case. Let the point  be the boundary of two different environments (see Figure 4.1). We would like to determine the way of the light from first environment to the second one.

The velocity of the light is constant for each environment. Therefore, it is determine by the formula

 (4.4)



Figure 4.2. Refraction of the light.

In our case the function



of the Lagrange problem does not depend from *x*. Therefore, we can us Lemma 4.1. Determine the equality (4.1)



Then we have

 (4.5)

where

.

The law of movement  can be found from the differential equation (4.5).

We know that the derivative is the tangent of the angle of the curve . Use the known formula



Then the equality (4.5) is transformed to



Using (4.4), we get the equality



This equality gives the relation between the angles of the sight and the refraction and the velocities of the light for different environments. This is the light refraction ***law of Snellius***.

### 4.5. Brachistochrone problem

We return to consider the ***Brachistochrone problem***. We would like to determine the curve *у* = *у*(*х*) from the origin of coordinates to the point with coordinates (*х*1,*у*1) such that the time of the movement by the influence of the weight only is minimal. We know (see Lecture 1), that the time of this movement can be described by the integral



We have the problem of its minimization with the boundary conditions

*у*(0) = 0, *у*(*х*1) = *у*1. (4.6)

The value under the integral does not depend from *x* directly, because we have



Therefore, we can use Lemma 4.1. Find the derivative



Then we have the equation (4.1)



where *c* is a constant. Hence, we obtain the equality



Therefore, we obtain



where  Then we get the equation



We have the equalities



Hence, we get



where *b* is constant. Make the substitution  in the first integral and  in the second integral. We obtain



Hence, we have the general solution of the Euler equation



Using first boundary condition (4.6) we get  Therefore, we have the equality

 (4.7)

where the constant *a* can be found from the second boundary condition (4.6). It depends from the concrete coordinates of the final point of the movement. The function that is described by the equation (4.7) is called the *cycloid*.

### Task 3. Euler equation. Special cases

We have two Lagrange problems with the functionals





where the functions *F* and *G* are given (see the Table). We have also the boundary conditions



Table of the values of the parameters.

|  |  |  |
| --- | --- | --- |
| variant |  |  |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |
| 5 |  |  |
| 6 |  |  |
| 7 |  |  |
| 8 |  |  |
| 9 |  |  |
| 10 |  |  |

The first problem is solved with using of the Euler equation. The second problem is solved with using of the first integral. For both problems it is necessary to choose the parameters  such that the corresponding Euler equation has a solution. Find this solution.

### Outcome

* The Euler equation is the necessary condition of the minimum for the Lagrange problem.
* This is a difficult enough second order differential equation.
* If we find the first integral of the system, we can transform the Euler equation to a first order differential equation.
* This is possible, if the function under the integral does not depend from the independent variable directly.
* The fall of the body, the refraction of the light and the brachistochrone problem are the practical applications of these results.

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### Next step

The problems of minimization for integral functionals have many practical applications. We know the general method of solving these problems. We will try to extend these results to other problems. At first, we will consider problems of functional minimization with many unknown functions.